On continuum modeling of sputter erosion under normal incidence: interplay between nonlocality and nonlinearity

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Under specific experimental circumstances, sputter erosion on semiconductor materials exhibits highly ordered hexagonal dot-like nanostructures. In a recent attempt to theoretically understand this pattern forming process, Facsko et al. [Phys. Rev. B **69**, 153412 (2004)] suggested a nonlocal, damped Kuramoto-Sivashinsky equation as a potential candidate for an adequate continuum model of this self-organizing process. In this study we theoretically investigate this proposal by (i) formally deriving such a nonlocal equation as minimal model from balance considerations, (ii) showing that it can be exactly mapped to a local, damped Kuramoto-Sivashinsky equation, and (iii) inspecting the consequences of the resulting non-stationary erosion dynamics.

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I. INTRODUCTION

Sputter erosion[1], the bombardment of solid target surfaces with ionized particles to remove or to detach target material, has a long tradition in physics as an experimental technique to clean, smooth, or appropriately prepare solid surfaces. The preparation of on the nanoscale perfectly flat surfaces, however, does not seem to be possible. This is because various surface roughening and smoothing processes compete during the sputtering process and can lead to self-organized pattern formation of the eroded surface morphology. Depending on the target material, temperature, ion beam energy, angle of incidence and various other parameters, one generically observes the development of rough surfaces, cellular patterns and, in particular under oblique incidence of the ion beam, the formation of more or less regular ripple patterns. For an overview on recent experimental results and theoretical approaches using continuum modeling we refer to Makeev et al.[2] and Valbusa et al.[3].

During the last five years, however, spectacularly novel experimental results have been reported by Facsko et al[4,5,6,7] showing that GaSb and InSb semiconductor targets eroded by Ar^+ ions under normal incidence can develop into a rather well ordered surface morphology with basically hexagonally arranged dot structures. Similar results have been subsequently reported by Gago et al[8] for Si targets under normal incidence and, more generally, by Frost et al.[9,10,11] for rotated InP, InSb and GaSb targets under oblique incidence (where as function of the inclination angle a variety of other patterns have also been observed). To theoretically explain the hexagonal ordering and taking advantage of earlier work by Elder et al.[12,13] (cf. also an even earlier study[14] on this subject), Facsko et al.[15] have recently suggested a (stochastically extended) stabilized or damped Kuramoto-Sivashinsky (KS) equation for the dynamics of the surface height H. This equation is given by

$$\partial_t H = -v_0 - \alpha H + \nu \nabla^2 H - D_{\text{eff}} \nabla^4 H + \frac{\lambda}{2}(\nabla H)^2 + \eta \quad (1)$$

and might be considered as a useful continuum model for the ion-beam erosion under normal incidence since it can successfully reproduce the experimentally observed hexagonally ordered dot-type structures. The physical origin of the six terms on the rhs of Eq.(1) is attributed to constant erosion velocity v_0, dissipation, effective surface tension (Bradley-Harper mechanism[16]), thermal and erosion induced diffusion, tilt dependent sputtering yield, and some stochasticity of the erosion process, respectively (cf. the original study[15] for details). Previous attempts[5,17] based on the (standard) KS equation, i.e. without the term $-\alpha H$, have reproduced cellular patterns without convincing evidence of a regular hexagonal ordering. As noted by Facsko et al. at the end of their paper[15], there are two basic problems with the applicability of Eq.(1) to erosion processes since, as Eq.(1) stands, it violates the translation invariance in the erosion direction and, moreover, the physical meaning of the dissipation term $-\alpha H$ is not directly obvious. Facsko et al.[15] briefly argued that (i) translational invariance can be restored by assuming that $-\alpha H$ has to be replaced by $-\alpha(H - \overline{H})$ with $\overline{H}$ being the erosion depth averaged over the sample area and (ii) that the term $-\alpha(H - \overline{H})$ might be interpreted as an approximation of a newly suggested redeposition effect of the sputtered target particles. Due to the nonlocal character of this term, however, the systematic connection between this nonlocally extended damped KS equation and Eq.(1) is far from being obvious and the main reason for our paper.

First, we argue on general grounds, by using balance considerations, symmetries and allowing for nonlocal dependencies, what the simplest functional form of the spatio-temporal dynamics of the morphology of ion sputtered surfaces under *normal* incidence of the ions might be, if nonlocal terms as suggested by Facsko et al.[15] are taken into account. Second, we show that the re-

sulting non-local model equation can be mapped to the damped KS equation by the use of a temporally nonlocal transformation. By that we show how the stabilized KS equation suggested by Facsko et al.[15] systematically fits into the theoretical framework of a continuum description of eroded surfaces obeying all desired invariances. Third, we discuss in detail the subtle interplay between potentially nonlocal terms in the evolution equation for the eroding surfaces and the role of the KS-nonlinearity $(\nabla H)^2$. Finally, we also investigate some consequences on the general evolution dynamics in these equations.

II. BALANCE EQUATION CONSIDERATIONS

The starting point of our investigation is the general, stochastically extended balance equation for the spatio-temporal evolution of the eroded surface morphology $H(\mathbf{x}, t)$ measured perpendicularly to the initially flat target surface with coordinates $\mathbf{x} = (x, y)$. Assuming that the particle density at the surface of the target material can be basically considered as being constant[18,19], such a balance equation reads quite generally[20]

$$\partial_t H = \nabla \cdot \mathbf{J}_H + F + \eta. \tag{2}$$

and expresses the fact that temporal changes of the erosion depth $H(\mathbf{x}, t)$ arise from two main contributions: (i) the detachment of target material leading to an in general inhomogenous and non-stationary, appropriately rescaled flux of eroded particles F (given by the number of eroded particles per time and surface area divided by the particle density) and (ii) the local rearrangements of the particles at the surface leading to a relaxational current $\mathbf{J}_H$ along the surface. Note that this balance only accounts for the target particles. The underlying driving of the erosion process, the flux of eroding ions I, enters indirectly into such a description: all terms on the right hand side of (2) depend on I in a way that they vanish if the sputtering process is turned off, $I = 0$. Note that thermally activated processes that do not depend on I and might lead to a further smoothing of the surface after the sputtering has been stopped are not taken into account. The spatio-temporal fluctuations $\eta = \eta(\mathbf{x}, t)$ entering in (2) mimic some stochasticity present in the erosion process and are usually assumed to be Gaussian white, i.e. having an average $\langle \eta(\mathbf{x}, t)\rangle_\eta = 0$ and a covariance $\langle \eta(\mathbf{x}, t)\eta(\mathbf{x}', t')\rangle_\eta = 2D\delta(\mathbf{x} - \mathbf{x}')\delta(t - t')$.

Considering periodic boundary conditions on an appropriately chosen, large enough sample area of the size L^2 and introducing the spatial average $\overline{...}(t) = (1/L^2)\int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dy...$ (being generally not equivalent to a stochastic average if taken on a finite area), the evolution of the mean erosion depth $\overline{H}(t)$ develops according to $\partial_t \overline{H}(t) = \overline{F} + \overline{\eta}$ which directly leads to

$$\overline{H}(t) - \overline{H}(0) = \int_0^t \overline{F}\, dt' + \int_0^t \overline{\eta}\, dt' \tag{3}$$

with $\overline{H}(0)$ being the spatial average of the initial condition $H(\mathbf{x}, 0)$ (which is usually set to zero under the assumption of an initially flat target surface). Already at this stage it is clear that the mean erosion depth $\overline{H}(t)$ is, in general, not a linear function in t implying a constant erosion velocity, but a rather complicated function that integrates over the history of the stochasticity and of the eroded flux.

In order to specify the admissible functional forms of the right hand side of Eq.(2), we reconsider the three fundamental symmetry requirements that are considered to be basic for the spatio-temporal evolution of surface morphologies[20]: (i) no dependence of (2) on the specific choice of the origin of time implying invariance of (2) under translation in time, (ii) no dependence of (2) on the specific choice of the origin of the coordinate system implying invariance of (2) under translation in the direction perpendicular to the erosion direction, and (iii) no dependence of (2) on the specific choice of the origin of the H-axis implying invariance of (2) under translation in growth direction. These symmetry requirements exclude any *explicit* dependence of $\nabla \cdot \mathbf{J}_H$ and F on the time t, the spatial position $\mathbf{x}$ and the erosion depth H, respectively. Following Facsko et al.'s argument[15], however, an implicite functional dependence on $H - \overline{H}$ and on the spatial derivatives of H is still admissable. Consequently, the detachment contribution in (2) is quite generally given by $F[\nabla H, H - \overline{H}]$ where $[\nabla H, ..]$ is the short hand notation for any derivative or combination of derivatives of H being compatible with the scalar character of F or H.

To proceed we (i) additionally apply invariance under rotation and reflection in the plane perpendicular to the erosion direction (which is suggested by the experimentally observed amorphorization of the target surface due to the erosion), (ii) expand the height dependent term f_β of F in a power series in $H - \overline{H}$, i.e. $f_\beta = \sum_n \beta_n (H - \overline{H})^n$, and (iii) perform a gradient expansion of the contribution f_α of F that solely contains derivatives of H. The latter implies that the lowest order terms (up to forth order in ∇ and second order in H) are given by $\nabla^2 H$, $(\nabla H)^2$, $\nabla^4 H$, $\nabla^2(\nabla^2 H)$, $(\nabla^2 H)^2)$, and $\nabla \cdot [(\nabla H)(\nabla^2 H)]$. Keeping only the lowest order contributions, the erosion flux is determined by

$$\begin{aligned} F \;=\; & F_0 \left[1 + \beta_1 (H - \overline{H}) + \alpha_1 \nabla^2 H + \alpha_2 (\nabla H)^2\right] \\ & + O\left(\nabla^4, H^2, (H - \overline{H})^2\right) \end{aligned} \tag{4}$$

with F_0 being a constant and negative since this part of the flux is antiparallel to the erosion direction. The lowest order possible mixing term between ∇H and $H - \overline{H}$ that could appear in (4) is given by $(H - \overline{H})\nabla^2 H$ and will be omitted since it can be considered as a higher order contribution to the term proportional to α_1 or β_1 in (4).

The functional form of the term representing the relaxational currents at the surface, $\nabla \cdot \mathbf{J}_H$ in (2) remains to be specified. Following Facsko et al.[15], we suppose

$\nabla \cdot \mathbf{J}_H = -D_{\text{eff}} \nabla^4 H$. This mimics the tendency of surface particles to reach energetically more favorable positions with a positive curvature $\nabla^2 H > 0$ and, therefore, leads to a current $\mathbf{J}_H \propto \nabla(\nabla^2 H)$.

Combining all ingredients and renaming the entering coefficients $b = \beta_1 F_0$, $a_1 = \alpha_1 F_0$, $a_2 = -D_{\text{eff}}$, and $a_3 = \alpha_2 F_0$, yields a minimal functional form for the evolution of the erosion depth under the afore-mentioned restrictions that is given by

$$\begin{aligned} \partial_t H &= a_1 \nabla^2 H + a_2 \nabla^4 H + a_3 (\nabla H)^2 \\ &\quad + b(H - \overline{H}) + F_0 + \eta. \end{aligned} \tag{5}$$

The functional form of Eq.(5) looks like the equation suggested by Facsko et al[15] that follows after their ad-hoc replacement of $-\alpha H$ by $-\alpha(H - \overline{H})$ in the damped KS equation. Note however, that the spatially constant part of the flux F, i.e. F_0, is, in general, not given by the mean constant erosion velocity since the latter is not constant during the course of the evolution.

Next we transform Eq.(5) into a coordinate system that moves with the mean erosion depth $\overline{H}(t)$,

$$h(\mathbf{x}, t) = H(\mathbf{x}, t) - \overline{H}(t) \tag{6}$$

with $h(\mathbf{x}, t)$ being the local erosion profile that obviously fulfills $\overline{h} = 0$ and $\partial_t \overline{h} = 0$ for all times t. The transformation (6) also guarantees that the invariance under translation in erosion direction $H \to H + z$ with z being an arbitrary constant (that only needs to hold in the coordinate system that is fixed in space) holds for *any* erosion profile $h(\mathbf{x}, t)$. As a consequence, the evolution dynamics for $h(\mathbf{x}, t)$ does not necessarily need to fulfill this requirement.

Using Eq.(5) and the facts that $\partial_t H = \partial_t \overline{H} + \partial_t h$ and that any terms consisting only of spatial derivatives of H have the same functional form in the comoving system when H is substituted by $\overline{H} + h$, the evolution equation for the erosion process in the comoving frame reads explicitly

$$\partial_t h = bh + a_1 \nabla^2 h + a_2 \nabla^4 h + a_3 (\nabla h)^2 + F_0 - \partial_t \overline{H} + \eta \tag{7}$$

Eq.(7) still constitutes a stochastic integro-partial differential equation by virtue of the nonlocal term $\partial_t \overline{H}$. Only if $\partial_t \overline{H} = F_0$ or, equivalently, the mean erosion depth $\overline{H}(t)$ were moving with a *constant* speed for all times, Eq.(7) would reduce to the damped stochastic KS equation. In general, however, this cannot be invoked, in particular because of the presence of the KS-type nonlinearity $(\nabla h)^2$. By taking the spatial average of Eq.(7), $a_3 \overline{(\nabla h)^2} + F_0 - \partial_t \overline{H} + \overline{\eta} = 0$, one can directly connect the mean erosion depth $\overline{H}(t)$ to the dynamics of the local erosion profile $h(\mathbf{x}, t)$,

$$\overline{H}(t) = F_0 t + \int_0^t dt' \left[a_3 \overline{(\nabla h)^2} + \overline{\eta} \right] \tag{8}$$

where $\overline{H}(0) = 0$ has been assumed. Consider first the deterministic part of (8), i.e. with $\overline{\eta} = 0$. Since the integrand of the second term on the right hand side of Eq.(8) is positive for all times, the mean erosion depth $\overline{H}(t)$ systematically deviates from a linear time evolution given by $F_0 t$. For a_3 being negative (positive) the mean erosion depth is therefore retarded (advanced) in comparison to $F_0 t$. Noteworthy, this fact is not a specific property triggered by the nonlocal term in Eq.(5), it is already present in the standard KS equation and from related studies there[18], it is known that the second term of the right hand side of (8) significantly contributes to the time evolution of $\overline{H}(t)$. The additional impact of the term $\int_0^t dt' \overline{\eta}$ is roughly that of a superposed Wiener process since η has been assumed to be a Gaussian and white.

III. TRANSFORMATION TO A LOCAL EQUATION

As explained in the previous section, it is not obvious how the nonlocal equations (5) and (7) are related to the damped KS equation. Here we show in a more general context how the nonlocal term can be elimated by an appropriate transformation. Specifically our statement is as follows: There is a transformation $h \to \hat{h}$ that maps the general form of a *nonlocal* stochastic evolution equation given by

$$\partial_t h = bh + G[\nabla h] + F_0 - \partial_t \overline{H} + \eta \tag{9}$$

to a *local* evolution equation in the transformed variables $\hat{h}$ reading

$$\partial_t \hat{h} = b\hat{h} + G[\nabla \hat{h}] + \eta \tag{10}$$

where, quite generally, the functional G in the Eqs.(9) and (10) can contain any combination of derivatives of h or $\hat{h}$, respectively, but no explicit dependence on h or $\hat{h}$.

As subsequently useful observation, we note that Eq.(10) is invariant under the transformation $\hat{h} \to \hat{h} + z \exp(bt)$ with z being an arbitrary constant. Obviously, (9) contains the model equation (7) as special case if we set $G[\nabla h] = a_1 \nabla^2 h + a_2 \nabla^4 h + a_3 (\nabla h)^2$.

To the constructive derivation of the statement: In order to find the desired transformation we use the ansatz

$$h \ \to \hat{h} = h + \frac{F_0}{b} + f(t) \tag{11}$$

where the time-dependent function $f(t)$ is so far arbitrary and will be subsequently determined. Since $\partial_t h = \partial_t \hat{h} - \partial_t f$ and spatial derivatives of h transform without any change to spatial derivatives in $\hat{h}$, insertion of (11) into Eq.(9) yields

$$\partial_t \hat{h} = b\hat{h} + G[\nabla \hat{h}] + \eta + \partial_t f - bf - \partial_t \overline{H}. \tag{12}$$

In order to arrive from Eq.(9) at Eq.(10), one has to demand that the last three terms on the right hand side of Eq.(12) vanish at any time,

$$\partial_t f = bf + \partial_t \overline{H}. \tag{13}$$

The linear nonautonomous ordinary differential equation (13) can be solved for $f(t)$ by standard means, e.g. by the method of variation of constants. Introducing $f(t) = c(t)\exp(bt)$ and solving the resulting equation $\partial_t c = \exp(-bt)\partial_t \overline{H}$ yields

$$f(t) = \mathrm{e}^{bt}\left[c(0) + \int_0^t dt'\,\mathrm{e}^{-bt'}\partial_{t'}\overline{H}(t')\right]. \qquad (14)$$

In general, the integration constant $c(0)$ in (14) can be arbitrarily chosen. This is a consequence of the afore-mentioned invariance of Eq.(10) with respect to $\hat{h} \to \hat{h} + \exp(bt)z$ (z arbitrary and constant) which implies that there is actually a continuous number of transformations leading from Eq.(9) to Eq.(10). A convenient choice, however, is to select as initial condition for $\hat{h}$ that $h(\mathbf{x},t)$ and $\hat{h}(\mathbf{x},t)$ initially coincide, i.e. $h(\mathbf{x},0) = \hat{h}(\mathbf{x},0)$. Consequently, $c(0) = -F_0/b$ holds and the transformation reads

$$h(\mathbf{x},t) = \hat{h}(\mathbf{x},t) - \frac{F_0}{b}(1-\mathrm{e}^{bt}) - \mathrm{e}^{bt}\int_0^t dt'\,\mathrm{e}^{-bt'}\partial_{t'}\overline{H}(t'). \qquad (15)$$

The transformation (15) that reduces the nonlocal equation (9) in h-system to a local equation (10) in $\hat{h}$-system has several specific properties. (i) It is a purely *temporal* (integral) transformation relating $h(\mathbf{x},t)$ in the physically meaningful coordinate system comoving to the mean evolution of the erosion depth with the evolution of $\hat{h}(\mathbf{x},t)$ in a temporally shifted system with no obvious physical significance. This shift $s(t) = h(\mathbf{x},t) - \hat{h}(\mathbf{x},t)$ being represented by the last two terms in (15), takes over the nonlocal properties of Eq.(9) and is therefore nonlocal by virtue of the last term in (15). Noteworthy, $s(t)$ integrates over the temporal history of the mean velocity of the erosion front via $\partial_{t'}\overline{H}(t')$. (ii) The stochastic part in Eq.(9) remains unchanged by the transformation. (iii) The transformation (15) is highly useful because it allows for the direct applicability of theoretical and numerical results obtained from Eq.(10) to Eq.(9). Note, however, that for a correct interpretation in the physical meaningful H- or h-system the full temporal information of the mean evolution of the erosion front $\overline{H}(t)$ needs to be separately determined. If the dynamics of $\hat{h}(\mathbf{x},t)$ is known, this can be achieved by using

$$\overline{H}(t) = F_0 t + \int_0^t dt'\,\left[\overline{G[\nabla\hat{h})]} + \overline{\eta}(t')\right]. \qquad (16)$$

(iv) The basic prerequisite for the transformation (15) is that Eq.(9) is linear in h. Extensions to nonlinear dependences on h in Eq.(9) do not seem to be generally feasible.

For the specific case under consideration, $G[\nabla h] = a_1\nabla^2 h + a_2\nabla^4 h + a_3(\nabla h)^2$, the transformation (15) can be further simplified. Using (6) it follows that

$$h(\mathbf{x},t) = \hat{h}(\mathbf{x},t) - a_3\int_0^t dt'\,\mathrm{e}^{-b(t'-t)}\overline{(\nabla h)^2}(t') \qquad (17)$$

and, consequently,

$$\partial_t\hat{h} = b\hat{h} + a_1\nabla^2\hat{h} + a_2\nabla^4\hat{h} + a_3(\nabla\hat{h})^2 + \eta \qquad (18)$$

which constitutes the damped KS equation in the $\hat{h}$-system and possesses the obvious invariance under the transformation $\{\hat{h}, a_3\} \to \{-\hat{h}, -a_3\}$. Eq.(17) shows the importance and the genuine interrelation of the KS-type nonlinearity for the non-triviality of the transformation. If a_3 equals zero, then simply $h(\mathbf{x},t) = \hat{h}(\mathbf{x},t)$ follows.

This argument can be generalized: Separating the functional $G[\nabla h]$ in Eq.(9) in terms that can be rewritten as the divergence of a flux, $G_{\mathrm{F}}[\nabla h] = \nabla\cdot\mathbf{j}_{\mathrm{F}}$, and terms $G_{\mathrm{NF}}[\nabla h]$ that cannot, the corresponding transformation is determined by $h(\mathbf{x},t) = \hat{h}(\mathbf{x},t) - \int_0^t dt'\,\mathrm{e}^{-b(t'-t)}\overline{G_{\mathrm{NF}}[\nabla h]}(t')$. Consequently, any term being nonlinear and not originating from a flux leads to analogous complications as the term $\overline{(\nabla h)^2}$. Only if $G_{\mathrm{NF}}[\nabla h] = 0$, then again $h(\mathbf{x},t) = \hat{h}(\mathbf{x},t)$ holds implying that $\overline{H}(t) = F_0 t$.

IV. SOME FURTHER RESULTS

A. Role of the nonlocal term

The general idea behind model equations such as Eq.(5) is that any individual term entering into the sum on the right hand side has its individual physical significance. As argued by Facsko et al.[15], the nonlocal term in (5) might be interpreted as a redepostion effect of eroded particles. To clarify the role of the nonlocal term in Eq.(5), $b(H-\overline{H})$, we disregard, for the moment, all terms in Eq.(5) that depend on derivatives of H and the stochastic fluctuations η. So, the equation $\partial_t H = F_0 + b(H-\overline{H})$ remains to be solved. Obviously $\partial_t\overline{H} = F_0$ holds and, consequently, (i) the mean depth evolution increases linearly with time for all times t according to $\overline{H} = F_0 t$ and (ii) $H - \overline{H}$ can be rewritten as $H - F_0 t$ showing that this term, individually considered, acts locally in the frame moving with $F_0 t$. The solution for the erosion profile H is then simply given by $H(\mathbf{x},t) = H(\mathbf{x},0)\exp(bt) + F_0 t$. For negative b, the impact of the term $b(H-\overline{H})$ is just an exponential diminishment of the initial target profile $H(\mathbf{x},0)$ in such a way that the overall shape and, in particular, the maxima, minima, and inclination points of $H(\mathbf{x},t) - F_0 t$ remain at same spatial positions. Only the amplitude of $H(\mathbf{x},t)$ decays in time. Consequently, it is not obvious whether the term $b(H-\overline{H})$ can be interpreted as a redeposition effect that should also lead to a lateral variation of the erosion profile.

B. Stability of a flat erosion front

Taking advantage of the afore-mentioned arguments on the role of the nonlinearity in (5), ignoring the stochastic

fluctuations for the moment and linearizing Eq.(5), i.e. omitting the term $(\nabla H)^2$, yields $\overline{H}(t) = F_0 t$. Therefore, a flat erosion front given by $H_{\mathrm{FF}}(\mathbf{x}, t) = F_0 t$ or, equivalently $h_{\mathrm{FF}}(\mathbf{x}, t) = 0$ solves the deterministic limit of the linearized version of Eq.(5) and determines its basic solution. Consequently, as far as the linear stability of the flat erosion front solution is concerned, the nonlocal term can be replaced by the local term $F_0 t$ in the linearized Eq.(5). Then, standard techniques can be invoked for the stability analysis of H_{FF}: Using an ansatz for a perturbation of $h_{\mathrm{FF}}(\mathbf{x}, t) = 0$ of the form $h \propto \exp[i\mathbf{k}\cdot\mathbf{x} + \sigma t]$, yields a dispersion relation $\sigma = b - a_1 k^2 + a_2 k^4$ $(k^2 = \mathbf{k}^2)$ for the growth rate $\sigma(k)$ of perturbations with a wave vector $\mathbf{k}$. Therefore, $\sigma(k)$ has its maximum at $k^2_{\max} = a_1/2a_2$ with $\sigma(k_{\max}) = b - a_1^2/4a_2$ and the flat erosion front H_{FF} is stable if $b \leq a_1^2/4a_2$.

C. Time dependence of the mean erosion depth $\overline{H}$

A crucial point of our discussion is the fact that variations of the mean erosion velocity about F_0, $\partial_t \overline{H} - F_0 = a_3 \overline{(\nabla H)^2} + \eta$, are generally non-zero and not even constant during the course of the erosion process. Here, we substantiate this by numerical simulations of Eq.(5) for a representative set of parameter values that leads to hexagonal patterns and draw some further conclusions.

In Fig. 1 we show the evolution of the ensemble or stochastically averaged evolution of $\partial_t \overline{H} - F_0$ given by $\partial_t \langle \overline{H} \rangle - F_0 = a_3 \langle \overline{(\nabla H)^2} \rangle$ since $\langle \overline{\eta} \rangle = \overline{\langle \eta \rangle} = 0$. For this quantity, the stochasticity necessarily present in individual runs of the erosion process, as shown in Fig. 2, is leveled out. In both cases, the initial target profile has been kept fixed to a small Gaussian distribution of the initial amplitudes with a maximum amplitude of $\eta = 0.01$ about the perfectly flat state.

The generic behavior as function of time consists of three parts: (i) For very small erosion times, $\partial_t \langle \overline{H} \rangle - F_0$ is very close to zero since here the nonlinearity in the evolution equation can be neglected and, therefore mainly the linear evolution of H contributes. Since $\partial_t \overline{H} - F_0$ is proportional to $a_3 \overline{(\nabla H)^2}$ this leads to a purely exponential increase for short times as depicted in the insets of Fig. 1. (ii) With increasing erosion time, the amplitude H eventually reaches values where the nonlinear term in Eq.(5) becomes comparable in size to linear terms. In this crossover time range, $\partial_t \langle \overline{H} \rangle - F_0$ increases very rapidly up to a point where the subsequent increase drastically slows down. (iii) For longer times, where the nonlinearity in Eq.(5) is fully developed, the KS-nonlinearity $(\nabla H)^2$ mainly reduces the further increase of $\partial_t \langle \overline{H} \rangle - F_0$ almost to a constant. Our numerical simulations that went very far into the full nonlinear regime, however, do not indicate a full saturation, but a slight systematic increase with time that depends almost linearly on t in the simulated time range. Similar behavior for individual erosion processes (albeit slightly modified by its intrinsic stochasticity) can also be read off from Fig. 2. An important consequence of this dynamical behavior of Eq.(5) is the fact that the underlying surface morphology $H(\mathbf{x}, t)$ does not saturate in a steady state unless $\partial_t \overline{H} - F_0$ fully saturates into a constant value. To substantiate this statement, we show in the insets of Fig. 2 for two representative erosion times the corresponding morphology behavior that clearly indicates a non-steady evolution.

Finally, we like to draw attention to a challenging point for further experiments. Due to the non-monotonic behavior of $\partial_t \langle \overline{H} \rangle - F_0$ as function of the erosion time that is triggered by the KS-nonlinearity, a specific experimental measurement of this quantity can lead to experimental evidence for the existence and dominance of no-flux terms as discussed in section III.

V. SUMMARY

To conclude, we stress the most important points of our investigation: (i) The nonlocal, damped KS equation suggested by Facsko et al.[15] can be obtained as minimal model from balance equation considerations if nonlocal terms compatible with the underlying invariances are allowed. Most importantly, such a nonlocal term being proportional to $H - \overline{H}$ leads in combination with the KS nonlinearity to a non-monotonic time evolution of the spatially averaged erosion depth and, connected with that, to a surface morphology that does not generally seem to approach a steady state (at least on the simulated time scales). (ii) The necessity for a non-local dependence arises from the fact that the KS nonlinearity cannot be rewritten in form of the divergence of a flux. If all terms entering into the evolution equation could be rewritten in form of such divergences, the non-local term could be replaced by a simple drift term $\overline{H} = F_0 t$. (iii) By using a specific transformation presented in section III, the nonlocal, damped KS equation can be exactly recast in form of a standard (non-local) damped KS equation. This property greatly simplifies the analysis of the evolution equations. Finally, we have also briefly proposed a simple experimental measurement in order to substantiate the pronounced effect of the KS nonlinearity or at least related non-flux nonlinearities during the erosion process. Building on the results presented here, we will discuss, in a subsequent publication, the rich pattern forming structure of an appropriately generalized non-local anisotropic damped KS equation for the case of oblique incidence.

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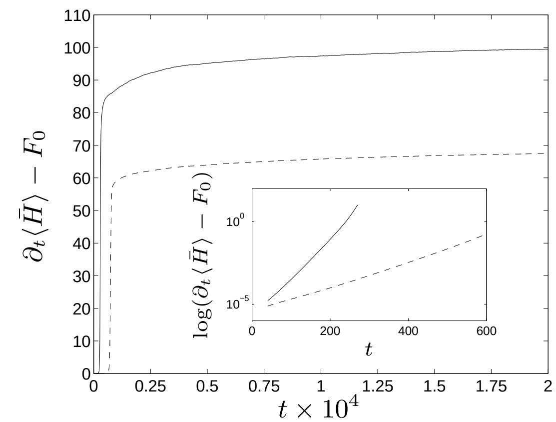

FIG. 1: Growth rate of the mean erosion depth calculated from Eq.(5), averaged over 100 runs. Parameters are the same as in the paper by Facsko et al.[15], except for a spatial step size of $dx = 1$ (mesh size 400×400,$b = -0.24$, $a_1 = -1$, $a_2 = -1$, $a_3 = 0.0025$, white noise with a maximum amplitude of $\eta = 0.01$). Solid line: $b = -0.22$, dotted line: $b = -0.24$. The inset shows a semilogarithmic plot of a portion of the data where linear terms determine the growth. A linear fit in this regime yields $\ln(\partial_t\langle\bar{H}\rangle - F_0) \propto 0.018 \cdot t$ for $b = -0.24$ and $\ln(\partial_t\langle\bar{H}\rangle - F_0) \propto 0.057 \cdot t$ for $b = -0.22$. This compares well to the result which can be obtained from the linearized version of Eq. (5), i.e. $2\sigma_{max} = 0.02$ for $b = -0.24$ and $2\sigma_{max} = 0.06$ for $b = -0.22$.

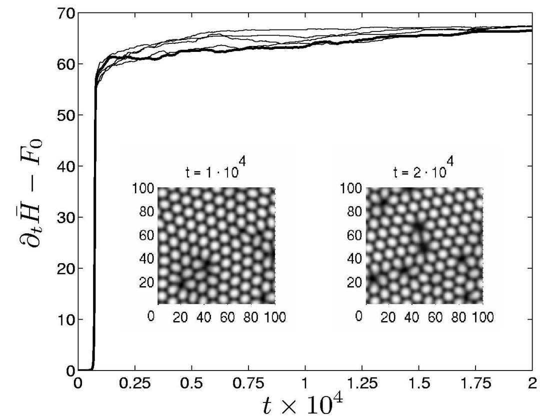

FIG. 2: Growth rate of the mean erosion depth calculated from Eq.(5) for five different noise sequences. Parameters are the same as in Fig. 1 with $b = -0.24$. Insets show a section of the surface morphology corresponding to the thick line in the figure at times $t = 1 \cdot 10^4$ and $t = 2 \cdot 10^4$.